Calculus
Epsilon, delta Proofs

Stewart book
page 101 15-20

15) $\lim_{x \to 1} 2x+3 = 5$

box work

$|2x+3-5| < \epsilon$

$|2x-2| < \epsilon$

$2|x-1| < \epsilon$

$|x-1| < \frac{\epsilon}{2}$

$0 < |x-1| < \delta$

Let $\delta = \frac{\epsilon}{2}$

then $0 < |x-1| < \delta$

$\Rightarrow 0 < 2|x-1| < 2\delta$

$\Rightarrow |2x-2| < 2\delta$

$|2x+3-5| < 2\delta$

since $\delta = \frac{\epsilon}{2}$

$|2x+3-5| < 2(\frac{\epsilon}{2})$

$\Rightarrow |2x+3-5| < \epsilon$ □

16) $\lim_{x \to -2} (\frac{1}{2}x+3) = 2$

box work

$|\frac{1}{2}x+3-2| < \epsilon$

$|\frac{1}{2}x+1| < \epsilon$

$\frac{1}{2}|x+2| < \epsilon$

$|x+2| < 2\epsilon$

$0 < |x-a| < \delta$

$0 < |x-(-2)| < \delta$

Let $\delta = 2\epsilon$

$0 < |x-(-2)| < \delta$

$\Rightarrow 0 < |x+2| < \delta$

$\Rightarrow 0 < \frac{1}{2}|x+2| < \frac{1}{2}\delta$

$\Rightarrow |\frac{1}{2}x+1| < \frac{1}{2}\delta$

$\Rightarrow |\frac{1}{2}x+3-2| < \frac{1}{2}\delta = \frac{1}{2}(2\epsilon) = \epsilon$

$\Rightarrow |\frac{1}{2}x+3-2| < \epsilon$ □

17) $\lim_{x \to -3} (1-4x) = 13$

boxwork

$|1-4x-13| < \epsilon$

$|-12-4x| < \epsilon$

$|(-4)(x+3)| < \epsilon$

$4|x+3| < \epsilon$

$|x+3| < \frac{1}{4}\epsilon$

$0 < |x-a| < \delta$

$0 < |x-(-3)| < \delta$

$0 < |x+3| < \delta$

Let $\delta = \frac{1}{4}\epsilon$

$0 < |x+3| < \delta$

$\Rightarrow 0 < 4|x+3| < 4\delta$

$\Rightarrow |4| \cdot |x+3| < 4\delta$

$\Rightarrow |-4| \cdot |x+3| < 4\delta$

$\Rightarrow |-4(x+3)| < 4\delta$

$\Rightarrow |-12-4x| < 4\delta$

$\Rightarrow |1-4x-13| < 4\delta = 4(\frac{1}{4}\epsilon) = \epsilon$

$\Rightarrow |1-4x-13| < \epsilon$ □

18) $\lim_{x \to 4} (7-3x) = -5$

box work

$$|7-3x-(-5)| < \epsilon$$
$$|7-3x+5| < \epsilon$$
$$|12-3x| < \epsilon$$
$$|(-3)(x-4)| < \epsilon$$
$$3|x-4| < \epsilon$$
$$|x-4| < \frac{\epsilon}{3}$$

Given: $0 < |x-4| < \delta$

let $\delta = \frac{\epsilon}{3}$

$$0 < |x-4| < \delta$$
$$\Rightarrow 0 < 3|x-4| < 3\delta$$
$$\Rightarrow |3| \cdot |x-4| < 3\delta$$
$$\Rightarrow |-3| \cdot |x-4| < 3\delta$$
$$\Rightarrow |(-3)(x-4)| < 3\delta$$
$$\Rightarrow |12-3x| < 3\delta$$
$$\Rightarrow |7-3x-(-5)| < 3\delta = 3\left(\frac{\epsilon}{3}\right) = \epsilon$$
$$\Rightarrow |7-3x-(-5)| < \epsilon$$

$\square$

19) $\lim_{x \to 3} \frac{x}{5} = \frac{3}{5}$

box work

$$\left|\frac{x}{5} - \frac{3}{5}\right| < \epsilon$$
$$\left|\frac{1}{5}(x-3)\right| < \epsilon$$
$$\frac{1}{5}|x-3| < \epsilon$$
$$|x-3| < 5\epsilon$$

Given: $0 < |x-3| < \delta$

Let $\delta = 5\epsilon$

$$0 < |x-3| < \delta$$
$$\Rightarrow 0 < \frac{1}{5}|x-3| < \frac{1}{5}\delta$$
$$0 < \left|\frac{x}{5} - \frac{3}{5}\right| < \frac{1}{5}\delta = \frac{1}{5}(5\epsilon) = \epsilon$$

$$\left|\frac{x}{5} - \frac{3}{5}\right| < \epsilon$$

$\square$

20) $\lim_{x \to 6} \left(\frac{x}{4} + 3\right) = \frac{9}{2}$

boxwork

$$\left|\frac{x}{4} + \frac{12}{4} - \frac{18}{4}\right| < \epsilon$$
$$\left|\frac{x-6}{4}\right| < \epsilon$$
$$\frac{1}{4}|x-6| < \epsilon$$
$$|x-6| < 4\epsilon$$

Given $0 < |x-6| < \delta$

Let $\delta = 4\epsilon$

$$0 < |x-6| < \delta$$
$$\Rightarrow 0 < \frac{1}{4}|x-6| < \frac{1}{4}\delta$$
$$\Rightarrow 0 < \left|\frac{x-6}{4}\right| < \frac{1}{4}\delta$$
$$0 < \left|\frac{1}{4}x - \frac{3}{2}\right| < \frac{1}{4}\delta$$
$$0 < \left|\frac{1}{4}x + 3 - \frac{9}{2}\right| < \frac{1}{4}\delta = \frac{1}{4}(4\epsilon) = \epsilon$$

$$\left|\frac{1}{4}x + 3 - \frac{9}{2}\right| < \epsilon$$

$\square$

21) $\lim_{x \to -5} \left(4 - \frac{3x}{5}\right) = 7$

box work

$$\left|\frac{20-3x}{5} - \frac{35}{5}\right| < \epsilon$$

$$\left|\frac{-3x-15}{5}\right| < \epsilon$$

$$\left|-\frac{3}{5}\right| |x+5| < \epsilon$$

$$\frac{3}{5}|x+5| < \epsilon$$

$$|x+5| < \frac{5}{3}\epsilon$$

$0 < |x-(-5)| < \delta$

Let $\delta = \frac{5}{3}\epsilon$

$0 < |x+5| < \delta$

$\Rightarrow \frac{3}{5}|x+5| < \frac{3}{5}\delta$

$\Rightarrow \left|-\frac{3}{5}\right| \cdot |x+5| < \frac{3}{5}\delta$

$\Rightarrow \left|\frac{-3-15}{5}\right| < \frac{3}{5}\delta$

$\Rightarrow \left|\frac{20-3x}{5} - \frac{35}{5}\right| < \frac{3}{5}\delta$

$\Rightarrow \left|4 - \frac{3x}{5} - 7\right| < \frac{3}{5}\delta = \frac{3}{5}\left(\frac{5}{3}\epsilon\right) = \epsilon$

$\left|4 - \frac{3x}{5} - 7\right| < \epsilon$ $\square$

22) $\lim_{x \to 3} \frac{x^2+x-12}{x-3} = 7$

$$\left|\frac{x^2+x-12}{x-3} - \frac{7(x-3)}{x-3}\right| < \epsilon$$

$$\left|\frac{x^2-6x+9}{x-3}\right| < \epsilon$$

$$\left|\frac{\cancel{(x-3)}(x-3)}{\cancel{(x-3)}}\right| < \epsilon$$

$$|x-3| < \epsilon$$

$0 < |x-3| < \delta$

Let $\delta = \epsilon$

$0 < |x-3| < \delta$

$\Rightarrow \left|\frac{x-3}{x-3}\right| \cdot |x-3| < \left|\frac{x-3}{x-3}\right|\delta$ since $x \neq 3$

$\Rightarrow \left|\frac{x^2-6x+9}{x-3}\right| < \left|\frac{x-3}{x-3}\right|\delta$

$\Rightarrow \left|\frac{x^2-x-12}{x-3} - \frac{7(x-3)}{(x-3)}\right| < \left|\frac{x-3}{x-3}\right|\delta$

$\Rightarrow \left|\frac{x^2-x-12}{x-3} - 7\right| < \delta$ since $x \neq 3$, $\frac{x-3}{x-3} = 1$

$\left|\frac{x^2-x-12}{x-3} - 7\right| < \epsilon$ $\square$

23) $\lim_{x \to a} x = a$

$|x-a| < \epsilon$

$0 < |x-a| < \delta$

Let $\delta = \epsilon$

$0 < |x-a| < \delta = \epsilon$

$|x-a| < \epsilon$ $\square$

24) $\lim_{x \to a} c = c$

$|c-c| < \epsilon$

$0 < \epsilon$

$0 < |x-a| < \delta$

Let $\delta = \epsilon$, $\epsilon > 0$

$0 < |x-a| < \delta = \epsilon$

$0 < |x-a| < \epsilon$

$\therefore 0 < \epsilon$ $\square$

25) $\lim_{x \to 0} x^2 = 0$

box work

$|x^2 - 0| < \epsilon$

$|x| \cdot |x| < \epsilon$

$|x| < 1$

$-1 < x < 1$

$\Rightarrow$

$0 < |x - 0| < \delta$

Let $\delta = \min\{1, \epsilon\}$

$0 < |x| < \delta$

if $|x| < 1$

$\Rightarrow -1 < x < 1$

$\Rightarrow x < 1$

$\Rightarrow |x| \cdot |x| < 1\delta$

$|x^2| < \delta = \epsilon$

$|x^2 - 0| < \epsilon$ $\square$

26) $\lim_{x \to 0} x^3 = 0$

$|x^3 - 0| < \epsilon$

$0 < |x - 0| < \delta$

Let $\delta = \min\{1, \epsilon\}$

$0 < |x| < \delta$

if $|x| < 1$

$\Rightarrow -1 < x < 1 \quad \Rightarrow |x^2| < 1$

$\Rightarrow |x||x^2| < 1\delta$

$|x^3 - 0| < \delta = \epsilon$

$|x^3 - 0| < \epsilon$

27) $\lim_{x \to 0} |x| = 0$

$||x| - 0| < \epsilon$

$||x|| < \epsilon$

$|x| < \epsilon$

$0 < |x - 0| < \delta$

Let $\delta = \epsilon$

$0 < |x - 0| < \delta$

$\Rightarrow |x| < \delta$

$||x|| < \delta$

$||x| - 0| < \delta$

$||x| - 0| < \epsilon$

$\square$

28) $\lim_{x \to 9^-} \sqrt[4]{9 - x} = 0$

$|\sqrt[4]{9-x} - 0| < \epsilon$

$|\sqrt[4]{9-x}| < \epsilon$

$\sqrt[4]{9-x} < \epsilon$

$9 - x < \epsilon^4$

$a - \delta < x < a$

$9 - \delta < x < 9$

$-\delta < x - 9 < 0$

$\delta > 9 - x > 0$

$0 < 9 - x < \delta$

Let $\delta = \epsilon^4$

$9 - x < \delta$

$\Rightarrow \sqrt[4]{9-x} < \sqrt[4]{\delta}$

$|\sqrt[4]{9-x}| < \sqrt[4]{\epsilon^4}$

$|\sqrt[4]{9-x}| < \epsilon$

$\square$

29) $\lim_{x\to 2} x^2-4x+5 = 1$

$|x^2-4x+5-1|<\epsilon$

$|x^2-4x+4|<\epsilon$

$|(x-2)(x-2)|<\epsilon$

$|x-2|\cdot|x-2|<\epsilon$

$0<|x-2|<\delta$

let $\delta = \min\{1, \epsilon\}$

let $|x-2|<1$

$|x-2|<\delta$

$\Rightarrow |x-2|\cdot|x-2| < 1\delta$

$\Rightarrow |(x-2)^2| < \delta$

$\Rightarrow |x^2-4x+4| < \delta$

$\Rightarrow |x^2-4x+5-1| < \delta = \epsilon$

$\Rightarrow |(x^2-4x+5)-1| < \epsilon$ $\square$

30) $\lim_{x\to 3} (x^2+x-4) = 8$

$|x^2+x-4-8|<\epsilon$

$|x^2+x-12|<\epsilon$

$|(x+4)(x-3)|<\epsilon$

$0<|x-3|<\delta$

let $\delta = \min\{1, \frac{\epsilon}{8}\}$

assume $|x-3|<1$

$\Rightarrow -1 < x-3 < 1$

$\Rightarrow 2 < x < 4$

$\Rightarrow |x+4| < |4+4| < 8$

since $|x-3|<\delta$

$\Rightarrow |x-3|\cdot|x+4| < 8\delta$

$|x^2+x-12| < 8(\frac{\epsilon}{8})$

$|(x^2+x-4)-8| < \epsilon$

$\square$

31) $\lim_{x\to -2} (x^2-1) = 3$

$|x^2-1-3|<\epsilon$

$|x^2-4|<\epsilon$

$|x-2|\cdot|x+2|<\epsilon$

$0<|x-(-2)|<\delta$

$0<|x+2|<\delta$

Let $\delta = \min\{1, \frac{\epsilon}{5}\}$

assume $|x+2|<1$

$\Rightarrow -1 < x+2 < 1$

$\Rightarrow -3 < x < -1$

$\Rightarrow -3-2 < x-2 \Rightarrow -5 < x-2 \Rightarrow |-5| > |x-2| \Rightarrow |x-2| < 5$

since $|x+2|<\delta$

$|x+2|\cdot|x-2| < 5\delta$

$|x^2-4| < 5(\frac{\epsilon}{5})$

$|x^2-1-3| < \epsilon$ $\square$

32) $\lim_{x \to 2} x^3 = 8$

$|x^3 - 8| < \epsilon$

$|(x-2)(x^2+2x+4)| < \epsilon$

$0 < |x-2| < \delta$

Let $\delta = \min\{1, \frac{\epsilon}{19}\}$

$|x-2| < 1$

$\Rightarrow -1 < x-2 < 1$

$\Rightarrow 1 < x < 3$

$\Rightarrow |x^2+2x+4| < |3^2 + 2(3)+4|$

$\Rightarrow |x^2+2x+4| < 19$

since $|x-2| < \delta$

$|x-2||x^2+2x+4| < 19\delta = 19(\frac{\epsilon}{19})$

$|x^3-8| < \epsilon$ $\square$